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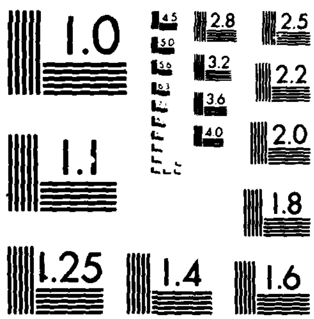

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ENHANCING OF SEMIGROUPS

by

Michael I. Taksar

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Fourth Floor, Encina Hall
Stanford University
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by

M. I. Taksar**

1. Introduction

1.1 Let D be a body in a three-dimensional space E and suppose that this body is heated at each point x to a certain temperature $h(x)$. Suppose that we observe the process of dissipation of heat and notice that the temperature decreases at each point x. The question is whether we can impose such boundary conditions that the original distribution of the temperature $h(x)$ is preserved. Physical intuition suggests the following solution. We have to look at those points of the boundary where the heat dissipates into outer space and put there reflectors which redistribute the heat over D proportionally to the rate of heat loss.

We shall show that the construction similar to the one suggested by physical intuition can be used in a more general situation.

1.2 In the situation described above let $f(t,x)$ stand for the temperature at the point x at time t. It is known that $f(t,x)$ can be obtained as a solution of the following system

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**Department of Operations Research, Stanford University.

$$\frac{\partial f}{\partial t} = Lf(t,\cdot) \quad , \tag{1.2.1}$$

$$Hf(t,\cdot) = 0 \quad , \tag{1.2.2}$$

$$f(0,x) = h(x) \quad .$$

Here L is an elliptic differential operator of the second order in the space E, and H is a linear operator in the space of functions on E (this operator corresponds to the boundary conditions; see Carslaw and Jaeger [1959]). Let $\mathscr{D}$ be a space of twice continuously differentiable functions g on E such that $Hg = 0$ and let Q_t be a semigroup whose generator coincides with L on $\mathscr{D}$. Then $f(t,x) = Q_t h(x)$. The inequality $f(t,x) \leq h(x)$ shows that $h(x)$ is an excessive function with respect to Q_t. This is equivalent to $\nu_t(dx) = h(x)dx$ being an excessive measure with respect to the conjugate semigroup T_t.

Placing reflectors at the boundary corresponds to changing the operator L in (1.2.1) into $L + \mathscr{L}$, where $\mathscr{L}$ is an integral operator, and changing boundary conditions, that is, replacing H in (1.2.2) by a new operator $\tilde{H}$ (the explicit expressions for $\mathscr{L}$ and $\tilde{H}$ in the one-dimensional case can be found in Feller [1952], Sections 18 and 22). Let $\tilde{Q}_t$ be the new semigroup corresponding to the solution of the new system and $\tilde{T}_t$ its conjugate. If under the new boundary conditions the temperature $h(x)$ is preserved, then ν is an invariant measure with respect to $\tilde{T}_t$. It is easy to see that for any function g on E

$$\tilde{T}_t g(x) \geq T_t g(x) \quad . \tag{1.2.3}$$

Under mild conditions in a more general situation we shall show that, given a semigroup T_t and an excessive with respect to T_t measure ν, one can find $\tilde{T}_t$ satisfying (1.2.3) for which ν is invariant.

If two semigroups $\tilde{T}_t$ and T_t satisfy (1.2.3) then we say that $\tilde{T}_t$ is larger than T_t, or $\tilde{T}_t$ is an enhancing of T_t.

We write $T_t = \tilde{T}_t$, a.e. μ, if for each function g $T_t g(x) = \tilde{T}_t g(x)$ for μ-almost every x.

In this paper we deal only with preserving positivity contraction normal semigroups, that is, semigroups T_t satisfying 1.2.A - 1.2.C below.

<u>1.2.A</u> If $g(x) \geq 0$ then $T_t g(x) \geq 0$ for each $t > 0$.

<u>1.2.B</u> For each $x \in D$

$$T_t 1(x) \leq 1 \quad \text{and} \quad \lim_{t \downarrow 0} T_t 1(x) = 1 .$$

<u>1.2.C</u> If $f(x_0) = 0$, then $T_0 f(x_0) = 0$.

With some abuse of notions we say that T_t is a semigroup on a Borel space D instead of saying that T_t is a semigroup in the Banach space of bounded measurable functions on D.

The same letter will be used for a measure and an integral with respect to this measure; thus, for P being a probability measure and ξ being a random variable, $P\xi$ (or $P\{\xi\}$) stands for the mathematical expectation of ξ.

Our main tool is the general theory developed in Taksar [1981]. We shall use notations and definitions of Taksar [1981] without reference.

2. Formulation of Results. Proof of Theorem of Existence

2.1 The main results of the paper are given by Theorems 1 and 2.

Theorem 1. Let T_t be a semigroup on a Borel space D such that

2.1.A $T_t 1(x)$ is a continuous function of t for every x.

If ν is a finite excessive (with respect to T_t) measure on D, then there exists a semigroup $\tilde{T}_t$ which is larger than T_t and for which ν is invariant.

Theorem 2. Suppose that T_t and ν satisfy the conditions of Theorem 1. If in addition ν is an extreme excessive measure, then $\tilde{T}_t$ is unique up to the measure ν.

To prove Theorem 1 we consider a Markov transition function $p(t,x;\Gamma)$ such that $T_t f(x) = p(t,x;f)$, the existence of which is proved in Dynkin [1965], Chapter 2, Theorem 2.1.

Then we consider a stationary Markov process generated by p and ν. We apply the main result of Taksar [1981] and construct a conservative, covering stationary Markov process. We show that the semigroup we are looking for corresponds to the transition function of the covering process.

2.2 We may assume without loss of generality that ν is a probability measure on D, i.e.,

(2.2.1) $$\nu(D) = 1 \quad .$$

Consider a stationary Markov process $(w(s),P)$, $s \in T =]-\infty,+\infty[$ with the state space D and the one-dimensional distribution ν. The existence of such a process was proved in Kuznecov [1973]. We can take the space W of all paths in D with random birth time α and death time β as a sample space of this process. (Note that $P\{W\}$ may be equal to infinity). The condition (2.2.1) is just the same as 1.2.A in Taksar [1981]. By virtue of the main result of Taksar [1981] there exists a process $(x_t,\bar{P})$ in a state space $D \cup V$, V being a singleton, for which $(w(s),P)$ is a subprocess in D. To construct the semigroup $\tilde{T}_t$ we have to find the transition function p of the process $(x_t,\bar{P})$.

Consider the entrance law ν_s (with respect to the transition function p) such that

$$\nu = \int_0^\infty \nu_s ds \quad .$$

The existence of this entrance law was proved in Dynkin [1980]. Let P_t^* be a Markov measure on W with transition function p and the one-dimensional distribution at time s equal to ν_{s-t} (we assume that $\nu_s \equiv 0$ for $s \leq 0$, therefore $P_t^*\{\alpha \neq t\} = 0$).

Let

$$\Pi(t;\, \Gamma) = P_t^*\{\beta \in \Gamma\} \quad , \quad \Gamma \subset T \quad ; \quad \Pi(\Gamma) = \Pi(0;\, \Gamma) \quad .$$

Consider an increasing process y_s on T with independent increments with the translation constant 0 and the Levy measure Π. (See

Taksar [1980] for definitions and a more detailed discussion). Let Q_y, $y \in T$ be the transition probabilities of this process and let $\sigma_\ell = \inf\{t:\ y_t > \ell\}$. For $\Gamma \subset D$, $x \in D$, $t > 0$ put

$$\begin{aligned} \bar{p}(t,x,\Gamma) &= p(t,x,\Gamma) + \int_0^t P_x\{\beta \in dy\}\, Q_y\{\int_0^{\sigma_t} P^*_{y_s}\{w(t) \in \Gamma\}ds\} \\ \bar{p}(t,x,V) &= 0 \quad . \end{aligned} \tag{2.2.2}$$

Here P_x is the transition probability of $(w(s);\ P)$.

<u>Theorem 2.2.1</u>. The kernel $\bar{p}$, given by (2.2.2), is a conservative transition function of the process $(x_t, \bar{P})$.

For the proof of the theorem we have to verify the following relations.

<u>2.2.A</u> For any $\Gamma,\ \Delta \subset D$

$$\bar{P}\{x_s \in \Gamma, x_t \in \Delta\} = \int_\Gamma \nu(dx)\bar{p}(t - s,x,\Delta) \quad . \tag{2.2.3}$$

<u>2.2.B</u> For each $x \in D$ and each $t > 0$

$$\bar{p}(t,x;\ D) = 1 \quad . \tag{2.2.4}$$

<u>2.2.C</u> For each $t,\ s > 0$ and $\Gamma \subset D$

$$\int_D \bar{p}(s,x;\ dy)\bar{p}(t,y;\ \Gamma) = \bar{p}(s + t,x;\ \Gamma) \quad . \tag{2.2.5}$$

The formula (2.2.3) is equivalent to the formula (5.7.3) of Taksar [1981], which was proved in the case of an extreme measure ν. However, similar

computations show that (5.7.3) of Taksar [1981] is true for the covering process $(x_t,\bar{P})$ constructed in Section 3 of Taksar [1981], with which we deal now.

To prove 2.2.B, we need

Lemma 2.2.2 Put

$$\varphi(z) = {}_z\int_0^{\sigma_t} P^*_{y_s}\{w(t) \in D\}ds \quad , \quad z \in T \quad .$$

Then $\varphi(z)$ is equal to 1 for all $z \in T$ except for a countable number of points.

Proof. Let $T^a =]a,\infty[$. Compute

$$(2.2.6) \quad \varphi(z) = Q_z \int_0^{\sigma_t} P^*_{y_s}\{\beta > t\}ds = Q_z \int_0^{\sigma_t} \Pi(y_s;\ T^t)ds = Q_0 \int_0^{\sigma_{t-z}} \Pi(y_s;\ T^{t-z})ds \quad .$$

Put $Y(u) = y_{\sigma_u}$, $Y(u-) = y_{\sigma_{u-}}$. Let λ_b be defined by formula (2.3) of Taksar [1980]. Put $u = t - z$. By virtue of Lemma 2.1 in Taksar [1980] the right side of (2.2.6) is equal to

$$Q_0\{Y(u-) < u, Y(u) > u\} \quad .$$

Since $Y(u-) \leq u$ and $Y(u) \geq u$, then

$$1 - \varphi(z) \leq Q_0\{Y(u-) = u\} + Q_0\{Y(u) = u\} \quad .$$

Let Λ_1 be the (countable) set of atoms of the measure λ_0; and Λ_2 be the (countable) set of atoms of the measure Π. Put $\Lambda = \Lambda_1 + \Lambda_2$.

By Lemma 2.1 of Taksar [1980]

$$
\begin{aligned}
Q_0\{Y(u) = u\} &= \int_0^u \Pi(x;\ \{u\})\lambda_0(dx) \\
&= \int_0^u \Pi\{u - x\}\lambda_0(dx) \qquad (2.2.7)\\
&= \sum_{\substack{x\in\Lambda_1 \\ x<u}} \lambda_0\{x\}\Pi\{u - x\} \quad .
\end{aligned}
$$

The right side of (2.2.7) differs from zero only for $u \in \Lambda$, therefore, for a countable number of u. Similarly

$$
Q_0\{Y(u-) = u\} = \lambda_0\{u\}\Pi\{T^0\} \quad . \qquad (2.2.8)
$$

The right side of (2.2.8) differs from zero only for $u \in \Lambda_1$. The lemma is proved.

Now show 2.2.B. By 2.1.A for each $u > 0$

$$
P_x\{\beta = u\} = \lim_{s\uparrow u} T_s1(x) - T_u1(x) = 0 \quad .
$$

Therefore, $P_x\{\varphi(\beta) \neq 1\} = 0$, and by (2.2.2)

$$
\begin{aligned}
\bar{p}(t,x,D) &= P_x\{\beta > t\} + P_x\{1_{\beta\leq t}\varphi(\beta)\} \\
&= P_x\{\beta > t\} + P_x\{\beta \leq t\} \\
&= 1 \quad .
\end{aligned}
$$

The following lemma is essential for the proof that $\bar{p}$ satisfies the Kolmogorov-Chapman equation.

<u>Lemma 2.2.3</u> For any $\Gamma \subset D$ and any $r < u < v$

$$Q_r \left\{ \int_0^{\sigma_u} ds \int_D P^*_{y_s}\{w(u) \in dz\} \int_0^{v-u} P_z\{\beta \in d\ell\} \cdot Q_\ell \left\{ \int_0^{\sigma_{v-u}} dm P^*_{y_m}\{w(v-u) \in \Gamma\} \right\} \right\} \tag{2.2.9}$$
$$= Q_r \int_{\sigma_u}^{\sigma_v} ds P^*_{y_s}\{w(v) \in \Gamma\} \quad .$$

<u>Proof</u>. Put

$$\psi_t(\ell) = Q_\ell \left\{ \int_0^{\sigma_t} P^*_{y_s}\{w(t) \in \Gamma\} ds \right\} .$$

Since P^*_r is a Markov measure with the transition probabilities P_z, then

$$\int_D P^*_r\{w(u) \in dz\} P_z\{\psi_{v-u}(\beta) 1_{\beta<v-u}\} \tag{2.2.10}$$
$$= P^*_r\{\psi_{v-u}(\beta - u) 1_{u<\beta<v}\} = P^*_r\{\psi_v(\beta) 1_{u<\beta<v}\} = P^*_r\{\psi'(\beta)\}$$

where $\psi'(x) = \psi_v(x) 1_{u<x<v}$. Taking into account the definition of the kernel $\Pi(x; -)$ and applying (2.2.10), we can rewrite the left side of (2.2.9) as

$$Q_r \left\{ \int_0^{\sigma_u} ds \int_u^v \Pi(y_s, d\ell) \psi_v(\ell) \right\} = Q_r \left\{ \int_0^{\sigma_u} ds \Pi(y_s; \psi') ds \right\}$$
$$= \int_r^u \lambda_r(dx) \Pi(x; \psi') \tag{2.2.11}$$
$$= \int_{-\infty}^{\infty} \lambda_r(dx) 1_{x<u} \Pi(x; \psi') \quad .$$

By Lemma 2.1 of Taksar [1980], (2.2.11) is equal to

$$Q_r \sum_{y_{t-}<y_t} 1_{y_{t-}<u}\, \psi'(y_t) = Q_r \sum_{y_{t-}<y_t} 1_{y_{t-}<u} 1_{u<y_t<v} \psi_v(y_t)$$

(2.2.12)

$$= Q_r\{\psi'(Y(u))\} \quad .$$

Using again Lemma 2.1 of Taksar [1980] and the strong Markov property of y_s, we get that (2.2.12) is equal to

$$Q_r\left\{1_{Y(u)<v}\left\{Q_{Y(u)} \int_0^{\sigma_v} P^*_{y_s}\{w(v) \in \Gamma\}ds\right\}\right\} = Q_r\left\{\int_{\sigma_u}^{\sigma_v} P^*_{y_s}\{w(v) \in \Gamma\}ds\right\} \quad ,$$

and the lemma is proved.

Now we are able to verify 2.2.C. Consider

$$\int_D \bar{p}(t,x;\, dy)\, \bar{p}(s,y;\, \Gamma)$$

$$= \int_D p(t,x;\, dy)\, p(s,y;\, \Gamma)$$

(2.2.13)

$$+ \int_D p(t,x;\, dy) \int_0^s P_y\{\beta \in du\}\, Q_u\left\{\int_0^{\sigma_s} P^*_{y_r}\{w(s) \in \Gamma\}dr\right\}$$

$$+ \int_0^t P_x\{\beta \in dy\}\, Q_y\left\{\int_0^{\sigma_t} dr \int_D P^*_{y_r}\{w(t) \in dz\}\, p(s,z;\, \Gamma)\right\}$$

$$+ \int_0^t P_x\{\beta \in dy\}\, Q_y\left\{\int_0^{\sigma_t} dr \int_D P^*_{y_r}\{w(t) \in dz\} \int_0^s P_z\{\beta \in d\ell\}\right.$$

$$\left. \cdot\, Q_\ell\left\{\int_0^{\sigma_s} P^*_{y_u}\{w(s) \in \Gamma\}du\right\}\right\} \quad .$$

By virtue of the Kolmogorov-Chapman equation for p the first term in the right side of (2.2.13) is equal to $p(s + t,x; \Gamma)$. Since $P^*_r\{w(s) \in \Gamma\} = P^*_{r+t}\{w(t + s) \in \Gamma\}$, then

$$Q_u\left\{\int_0^{\sigma_s} P^*_{y_r}\{w(s) \in \Gamma\}\, dr\right\} = Q_{u+t}\left\{\int_0^{\sigma_{s+t}} P^*_{y_r}\{w(t + s) \in \Gamma\}\, dr\right\} .$$

Together with the Markov property of $(w(\cdot),P)$ this relation yields that the second term in the right side of (2.2.13) is equal to

$$(2.2.14) \qquad \int_t^{t+s} P_x\{\beta \in dy\}\, Q_y\left\{\int_0^{\sigma_{t+s}} P^*_{y_s}\{w(t + s) \in \Gamma\}\, dr\right\} .$$

Since P^*_u is a Markov measure with the transition function p, then the third term in the right side of (2.2.13) is equal to

$$(2.2.15) \qquad \int_0^t P_x\{\beta \in dy\}\, Q_y\left\{\int_0^{\sigma_t} P^*_{y_r}\{w(t + s) \in \Gamma\}\, dr\right\} .$$

By Lemma 2.2.3 the fourth term in the right side of (2.2.13) is equal to

$$(2.2.16) \qquad \int_0^t P_x\{\beta \in dy\}\, Q_y\left\{\int_{\sigma_t}^{\sigma_{s+t}} P^*_{y_r}\{w(t + s) \in \Gamma\}\, dr\right\} .$$

Adding (2.2.14), (2.2.15) and (2.2.16) we get that (2.2.13) reduces to

$$p(s + t,x; \Gamma) + \int_0^{t+s} P_x\{\beta \in dy\}\, Q_y\left\{\int_0^{\sigma_{t+s}} P^*_{y_r}\{w(s + t) \in \Gamma\}\, dr\right\}$$

$$= \bar{p}(s + t,x; \Gamma) \quad ,$$

and that completes the proof of Theorem 2.2.1.

2.3 Put

$$\tilde{T}_t f(x) = \int_D \bar{p}(t,x;\, dy)\, f(y) \quad .$$

By Theorem 2.2.1 $\bar{p}$ satisfies 2.2.A - 2.2.C; therefore, T_t is a contraction semigroup such that $\tilde{T}_t 1 \equiv 1$. It is obvious that $\tilde{T}_t$ is an enhancing of T_t. Inasmuch as $\bar{p}$ is the transition function of a stationary Markov process with the one-dimensional distribution ν, then ν is an invariant measure with respect to $\tilde{T}_t$.

Remark. Although the state space of the covering process $(x_t, \bar{P})$ is larger than D, we were able to exclude the additional point V from the formulation of the final result. To this end we proved that the transition function of $(x_t, \bar{P})$ restricted to D remains a transition function. That was possible due to the condition 2.1.A. In a general situation, when 2.1.A is not satisfied, one can prove a weaker analogue of Theorem 1 in which $\tilde{T}_t$ is not a semigroup on the same space D but on a larger space $E \supset D$.

3. Theorem of Uniqueness

3.1 Suppose $\tilde{T}_t$ is constructed and we want to prove that $\tilde{T}_t$ is unique up to the measure ν. The main idea of the proof is to apply Theorem 2 of Taksar [1981]. To this end we have to construct a stationary Markov process $(x_t, \bar{P})$ with the one-dimensional distribution ν and the transition function $\bar{p}$ (the function $\bar{p}$ generates the semigroup $\tilde{T}_t$). That must be done in such a way that the subprocess in D of $(x_t, \bar{P})$ is $(w(s), P)$.

The general outline of the construction is the following. We construct $(x_t,\bar{P})$ and apply Theorem 9.3 of Dynkin [1965] to the transition functions p and $\bar{p}$. There exists a multicative functional α_t such that $p(s,x;\Gamma) = \bar{P}_x\{1_\Gamma(x_s)\alpha_s\}$, where $\bar{P}_x$ are the transition probabilities of x_t. If α_t took on only 0 and 1 values that would be almost the end of the construction. In this case we would consider a family of stopping times $\sigma_s = s + \inf\{t\colon\ \theta_s\alpha_{t-s} = 0\}$ (θ_s is a shift operator in the sample space of x_t). The family σ_s would have the same properties as the family of hitting times of a set in the state space. We would then put $x_t^* \in V$ if $\sigma_s = t$ for some $s < t$ and $x_t^* = x_t$ otherwise. The process $(x_t^*,\bar{P})$ would have the same finite dimensional distributions as $(x_t,\bar{P})$, and its subprocess in D would be $(w(s),P)$.

Unfortunately $\alpha_t(\omega)$ may take on any values between zero and one, and it is necessary to use a coupling technique to overcome this difficulty. We consider a new sample space $\tilde{\Omega} = \Omega \times (T)^\infty$ and a probability measure Q on $\tilde{\Omega}$ with marginal distribution on Ω equal to $\bar{P}$. A family of random variables $\tau_s(\tilde{\omega})$ is constructed in such a way that the conditional probability of τ_s being greater than t given ω is equal to $\alpha_{t-s}(\theta_s\omega)$. The family τ_s has all the properties that σ_s has, except τ_s is not measurable with respect to the σ-field generated by x_t. Nevertheless, it is possible to construct x_t^*, acting in the same way as if τ_s was the first hitting time after s of a set in the state space. In this case x_t^* has the same finite dimensional distributions as x_t and the subprocess of x_t^* in D is equal to $(w(s),P)$. Therefore, we may apply Theorem 2 of Taksar [1981] to obtain the final result.

3.2 Let Ω be the space of all paths $\omega(t)$ in D, $-\infty < t < +\infty$. Suppose that ν and T_t satisfy the conditions of Theorem 2 and we have constructed $\tilde{T}_t$ which is an enhancing of T_t and with respect to which ν is invariant. Let $\bar{p}(t,x;\Gamma)$ be a transition function such that $\tilde{T}_t f(x) = \bar{p}(t,x; f)$. By the Kolmogorov Theorem there exists a stationary Markov measure $\bar{P}$ on Ω with the one-dimensional distribution ν and the transition function $\bar{p}$. Let θ_t be the shift operator in the space Ω, that is $\theta_t\omega(s) = \omega(s + t)$. Put $x_t(\omega) = \omega(t)$ and put $\mathscr{F}_t = \sigma(x_s, s < t)$, $\mathscr{F}^t = \sigma(x_s, s > t)$. Let $\bar{P}_x$ be the transition probabilities of $(x_s, \bar{P})$, that is $\bar{P}_x$ is a measure of $\mathscr{F}^0$ such that

$$\bar{P}_x\{x_{t_1} \in dx_1, \ldots, x_{t_n} \in dx_n\}$$

$$= \bar{p}(t_1,x; dx_1)\, \bar{p}(t_2 - t_1,x_1; dx_2) \cdots \bar{p}(t_n - t_{n-1},x_{n-1}; dx_n) \quad ,$$

$$0 < t_1 < t_2 \cdots t_n \quad .$$

By Theorem 9.3 of Dynkin [1965] there exists an almost homogenous, almost multiplicative functional (AHAMF) $\bar{\alpha}_t(\omega)$ such that

$$p(x,t; \Gamma) = \bar{P}_x\{\bar{\alpha}_t(\omega)\, 1_\Gamma(x_t)\} \quad . \tag{3.2.1}$$

By properties of AHAMF the random variable $\bar{\alpha}_t$ is $\sigma(x_s, 0 \le s \le t)$ - measurable. Formula (9.13) in Dynkin [1965] shows that for any t and for all $x \in D$

$$\bar{\alpha}_t \le 1 \qquad \text{a.s. } \bar{P}_x \quad .$$

<u>Lemma 3.2.1</u> There exists AHAMF α_t such that for all $x \in D$ a.s. $\bar{P}_x$:

3.2.α. For any t, $\alpha_t = \bar{\alpha}_t$.

3.2.β. α_t is a right-continuous function of t, $t > 0$, and $\alpha_{0+} = 1$.

<u>Proof</u>. Since $\bar{\alpha}_t \leq 1$ a.s. $\bar{P}_x$, then $\bar{\alpha}_t \leq \bar{\alpha}_s$ a.s. P_x for $s < t$. Let R-lim stand for the limit taken over rational points and put

$$\alpha_t(\omega) = \text{R-}\limsup_{r \downarrow t} \bar{\alpha}_r(\omega) \quad .$$

Inasmuch as $\bar{\alpha}_r$, r-rational, is a decreasing function of r a.s. $\bar{P}_x$, then α_t is right-continuous. It is obvious that

$$\alpha_t \leq \bar{\alpha}_t \quad \text{a.s.} \quad \bar{P}_x \quad . \tag{3.2.2}$$

By 1.2.B and the Kolmogorov-Chapman equation for p we get

$$p(t,x;\, D) \uparrow p(s,x;\, D) \quad \text{as} \quad t \downarrow s \quad . \tag{3.2.3}$$

By the monotone convergence theorem we have

$$\text{R-}\lim_{r \downarrow t} \bar{P}_x\{\bar{\alpha}_r(\omega)\} = \bar{P}_x\{\bar{\alpha}_t(\omega)\} \quad . \tag{3.2.4}$$

Combining (3.2.1), (3.2.2), (3.2.3) and (3.2.4) we get that $\alpha_t = \bar{\alpha}_t$ a.s. $\bar{P}_x$. By 1.2.B

$$\lim_{s \downarrow 0} p(s,x;\, D) = 1 \quad ,$$

therefore, $\lim_{s\downarrow 0} \alpha_s = 1$ a.s. $\bar{P}_x$ and the lemma is proved.

Note that because ν is an extreme excessive measure then it has to be null-excessive (see [3]), and that implies for ν a.e. x

$$p(s,x;\ D) \to 0 \quad \text{as} \quad s \to \infty \quad .$$

(We exclude the trivial case in which ν is an invariant measure). Therefore, for ν a.e. x

(3.2.5) $$\lim_{t\to\infty} \alpha_t(\omega) = 0 \quad \text{a.s.} \quad \bar{P}_x \quad .$$

Put

$$\alpha_t^s(\omega) = \alpha_{t-s}(\theta_s \omega)$$

and let $\alpha^s(dy|\omega)$ stand for the measure on T with the distribution function $F(\cdot)$ equal to $1 - \alpha_\cdot^s(\omega)$. Inasmuch as (3.2.5) holds, $1 - \alpha_\cdot^s(\omega)$ is a proper distribution function for $\bar{P}$ - almost all ω.

Let $\tilde{\Omega}$ be a product of Ω and $(T)^\infty$ and the measurable structure in $\tilde{\Omega}$ be gererated by the product of the corresponding measurable structures. We denote by the same letter $\mathscr{F}$ the σ-field in $\tilde{\Omega}$ generated by the sets of the type $A \times (T)^\infty$ where $A \in \mathscr{F} \equiv \mathscr{F}_\infty$.

The following theorem is most important in carrying out the construction of the process x_t^*.

<u>Theorem 3.2.2</u>. There exists a probability measure Q on $\tilde{\Omega}$ and a family of random variables $\tau_s(\tilde{\omega})$ such that

3.2.A For each $A \in \mathscr{F}$

$$Q\{A \times (T)^{\infty}\} = \bar{P}\{A\} \quad .$$

3.2.B For each $s \in T$ and $\Gamma \subset T$

$$Q\{\tau_s(\tilde{\omega}) \in \Gamma | \mathscr{F}\} = \alpha^s(\Gamma|\omega) \quad \text{a.s.} \quad \bar{P} \quad .$$

3.2.C For any fixed $s < t$ $\tau_s < \tau_t$; and

$$\tau_s = \tau_t \quad \text{a.s.} \quad Q \quad \text{on the set} \quad \{\tau_s > t\} \quad .$$

3.2.D If $s < u \leq t < r$ then the events $\{\tau_s < u\}$ and $\{\tau_t > r\}$ are conditionally independent given $\mathscr{F}$.

Proof.

1) Put $\tilde{\Omega}^0 = \Omega, \tilde{\Omega}^{k+1} = \tilde{\Omega}^k \times T$. The set $\tilde{\Omega}^k$ consists of all $k+1$-tuples $(\omega, t_1 t_2, \ldots, t_k)$, where $\omega \in \Omega, t_k \in T$. To construct the measure Q on $\tilde{\Omega}$ it is enough to construct a sequence of kernels $n_1, n_2, \ldots, n_k, \ldots$ where n_k is a kernel from $\tilde{\Omega}^{k-1}$ into T and then put

$$(3.2.6) \quad \begin{aligned} &Q\{d\omega \times dt_1 \times dt_2 \times \cdots \times dt_m \times T \times T \times \cdots\} \\ &= \bar{P}\{d\omega\} n_1(\omega; dt_1) n_2(\omega, t_1; dt_2) \cdots n_m(\omega, t_1, t_2, \ldots, t_{m-1}; dt_m) \quad . \end{aligned}$$

Let $r_1, r_2, \ldots, r_k, \ldots$ be a sequence of all rational numbers. For typographical purposes we write r_k and $r(k)$ interchangeably. The last coordinate in $\tilde{\omega}^k$ corresponds to the value of $\tau_{r(k)}$, i.e., $\tau_{r(k)}(\tilde{\omega}) = t_k$. We shall construct n_k by induction in such a way that 3.2.B - 3.2.C will hold for the family $\{\tau_{r(1)}, \tau_{r(2)}, \ldots, \tau_{r(k)}\}$ if

these properties are satisfied for the family $\{\tau_{r(1)},\ldots,\tau_{r(k-1)}\}$. (Note that due to (3.2.6) the property 3.2.A will hold automatically.)

2) By the properties of AHAMF and definitions of $\alpha_t^s(\omega)$

$$\alpha_t^s \ \alpha_u^t = \alpha_u^s \quad \text{a.s.} \ \bar{P} \tag{3.2.7}$$

for any fixed $s < t < u$. Therefore, (3.2.7) is true for all rational s, t, and u simultaneously. Since α_u^t is right-continuous in u (3.2.7) is true a.s. $\bar{P}$ for any rational s and t and all u. Put $n_1(\omega;\ \Gamma) = \alpha^{r(1)}(\Gamma|\omega)$ and put $\tau_{r(1)}(\tilde{\omega}) = t_1$. The properties 3.2.B - 3.2.D are satisfied trivially for the family consisting of a single element $\tau_{r(1)}$.

Suppose that the kernels $n_1, n_2,\ldots,n_{k-1}$ are constructed in such a way that the family of random variables $(\tau_{r(1)} = t_1, \tau_{r(2)} = t_2, \ldots, \tau_{r(k-1)} = t_{k-1})$ satisfies 3.2.B - 3.2.D. Let $\Lambda_m = \{r_1, r_2, \ldots, r_m\}$ and $b = \inf \Lambda_{k-1} \cap [r_k, +\infty[$, $a = \sup \Lambda_{k-1} \cap]-\infty, r_k]$. Suppose that $a > -\infty$ and $b < +\infty$; consequently there exists $i \le n - 1$ and $j \le n - 1$ such that $a = r_i$, $b = r_j$. Put

$$\begin{aligned} & n_k(\omega, t_1, t_2, \ldots, t_k;\ \Gamma) \\ &= n_k(\omega, t_i, t_j;\ \Gamma) \\ &= \begin{cases} 1_{t_i}(\Gamma) & , \text{ if } t_i > r_k \ ; \\ \alpha^{r(k)}(\Gamma'|\omega) + \alpha_b^{r(k)} 1_{t_j}(\Gamma'') & , \text{ if } t_i \le r_k \ . \end{cases} \end{aligned} \tag{3.2.8}$$

Here $\Gamma' = \Gamma \cap [r_k, b]$, $\Gamma'' = \Gamma \cap]b, \infty[$.

3) We have to check that the properties 3.2.B - 3.2.D hold for the new family $\{\tau_{r(1)},\ldots,\tau_{r(k-1)},\tau_{r(k)}\}$. The property 3.2.C is trivially satisfied by the construction of the kernel (3.2.8).

To check 3.2.B we may consider only sets Γ lying on the ray $[r_k,\infty[$. (In the following formulae conditioning with respect to $\mathcal{F}$ is assumed; and all the equalities are satisfied a.s. $\mathcal{P}$).

$$Q\{t_k \in \Gamma\} = Q\{t_k \in \Gamma,\ t_i > r_k\} + Q\{t_k \in \Gamma,\ t_i \le r_k\}$$

$$(3.2.9) \qquad = Q\{t_i \in \Gamma\} + Q\{t_k \in \Gamma | t_i \le r_k\}\ Q\{t_i \le r_k\}$$

$$= \alpha^a(\Gamma) + (1 - \alpha^a_{r(k)})(\alpha^{r(k)}(\Gamma') + \alpha^{r(k)}_b\ \alpha^b(\Gamma'')) \quad .$$

The last equality in (3.2.9) is due to the conditional independence of $\tau_a = t_i$ and $\tau_b = t_j$ given $\mathcal{F}$ on the set $\{t_i < b\}$ (property 3.2.D). By (3.2.7)

$$(3.2.10) \qquad \alpha^a(\Gamma) = \alpha^a_{r(k)}\alpha^{r(k)}(\Gamma)$$

and

$$(3.2.11) \qquad \alpha^{r(k)}_b\alpha^b(\Gamma'') = \alpha^{r(k)}(\Gamma'') \quad .$$

Comparing (3.2.10) and (3.2.11) with the right side of (3.2.9), we get that (3.2.9) is equal to $\alpha^{r(k)}(\Gamma)$.

4) Now prove 3.2.D. Consider the case $t = r(k)$, $s = r(i) = a$ and $u < t < r$. Suppose $r < b$. Then

$$Q\{t_i < u, t_k > r\} = Q\{t_i < u, r < t_k \leq b\} + Q\{t_i < u, t_k > b\}$$

$$= Q\{t_i < u\}(\alpha_r^{r(k)} - \alpha_b^{r(k)}) + Q\{t_i < u\}\alpha_b^{r(k)}$$

$$= Q\{t_i < u\}\alpha_r^{r(k)}$$

$$= Q\{t_i < u\}\ Q\{t_k > r\} \quad .$$

All the other cases are considered similarly.

5) We have constructed a measure Q on $\tilde{\Omega}$ and a family $\tau_s(\tilde{\omega})$ (s-rational) which satisfy 3.2.A - 3.2.D. Show that τ_s is right-continuous in s. Since s takes on only a countable number of values it is sufficient to show that for any fixed r $\tau_u \downarrow \tau_r$ a.s. Q when $u \downarrow r$. Owing to the fact that $\{\tau_r \neq \tau_u\} \subset \{\tau_r \leq u\}$ a.s. Q, we get

$$(3.2.12) \quad Q\{\tau_u - \tau_r > u - r\} \leq Q\{\tau_u \neq \tau_r\} = Q\{\tau_r \leq u\} = \bar{P}\{1 - \alpha_u^r\} \quad .$$

Since $\alpha_u^r(\omega) = \alpha_{u-r}(\theta_r\omega)$ then

$$(3.2.13) \qquad \lim_{u \downarrow r} \alpha_u^r = \lim_{\varepsilon \downarrow 0} \alpha_\varepsilon(\theta_r\omega) = 1 \quad \text{a.s. } \bar{P} \quad ,$$

(see 3.2.β). Therefore, the limit in the right side of (3.2.12) is equal to zero whenever $u \downarrow r$. For irrational u put

$$(3.2.14) \qquad \tau_u(\tilde{\omega}) = \text{R-}\limsup_{s \downarrow u} \tau_s(\tilde{\omega}) \quad .$$

Since a.s. Q τ_s is an increasing right-continuous function of s for rational s, then the right side of (3.2.14) has limit for all u a.s. Q.

Therefore,

$$Q\{\tau_u \geq t \mid \mathscr{F}\} = \operatorname*{R-lim}_{s\downarrow u} Q\{\tau_s \geq t \mid \mathscr{F}\}$$

$$= \operatorname*{R-lim}_{s\downarrow u} (Q\{\tau_s > t \mid \mathscr{F}\} + Q\{\tau_s = t \mid \mathscr{F}\})$$

$$= \operatorname*{R-lim}_{s\downarrow u} (\alpha_t^s + (\alpha_{t-}^s - \alpha_t^s)) \quad .$$

By virtue of 2.1.A $\alpha_t = \alpha_{t-}$ a.s. $\bar{P}$ for each fixed t. Consequently, $\alpha_{t-}^s = \alpha_t^s$ a.s. $\bar{P}$ and

$$(3.2.15) \qquad Q\{\tau_u \geq t \mid \mathscr{F}\} = \operatorname*{R-lim}_{s\downarrow u} \alpha_t^s = \operatorname*{R-lim}_{s\downarrow u} (\alpha_t^u/\alpha_s^u) = \alpha_t^u \quad .$$

The last equality in (3.2.15) is due to (3.2.13). Applying right-continuity of α_t^u in t, we get

$$Q\{\tau_u > t \mid \mathscr{F}\} = \alpha_{t+}^u = \alpha_t^u \quad .$$

Therefore, 3.2.B holds for all real s. To verify 3.2.C and 3.2.D one has to pass to the limit in the corresponding relations for rational s and t.

<u>Lemma 3.2.3</u> Let $\mathscr{C}_t$ be the σ-field in $\tilde{\Omega}$ generated by all random variables of the form $g(\tau_s)$, where $s < t$ and g is a measurable function with support on $[-\infty,t]$. Then for each t the random variable τ_t and the σ-field $\mathscr{C}_t \vee \mathscr{F}_t$ are conditionally independent given x_t.

<u>Proof</u>. Let ξ be $\mathscr{F}_t$-measurable and $s_1 < s_2 < \cdots < s_k = t$.

Let $g_i(x) = 1_{]u_i, v_i]}$, where $s_i \leq u_i < v_i \leq s_{i+1}$, $i = 1, 2, \ldots, k$ (we assume $s_{k+1} = \infty$). By 3.2.B and 3.2.D

$$
\begin{aligned}
& Q\left\{\xi g_1(\tau_{s_1}) g_2(\tau_{s_2}) \cdots g_k(\tau_t) \mid x_t\right\} \\
(3.2.16) \qquad & = Q\left\{Q\left\{\xi g_1(\tau_{s_1}) \cdots g_k(\tau_t) \mid \mathscr{F}\right\} \mid x_t\right\} \\
& = Q\left\{\xi\left(\alpha_{u_1}^{s_1} - \alpha_{v_1}^{s_1}\right) \cdots \left(\alpha_{u_k}^{t} - \alpha_{v_k}^{t}\right) \mid x_t\right\} .
\end{aligned}
$$

Because $\alpha_{u_i}^{s_i} - \alpha_{v_i}^{s_i}$ is $\mathscr{F}_{v_i}$-measurable and $\alpha_{u_k}^{t} - \alpha_{v_k}^{t}$ is $\mathscr{F}^t$-measurable, then by the Markov property for $(x_t, \bar{P})$ the right side of (3.2.16) is equal to

$$
\begin{aligned}
& Q\left\{\xi \prod_{i=1}^{k-1} \left(\alpha_{u_i}^{s_i} - \alpha_{u_i}^{s_i}\right) \mid x_t\right\} Q\left\{\alpha_{u_k}^{t} - \alpha_{v_k}^{t} \mid x_t\right\} \\
(3.2.17) \qquad & = Q\left\{\xi \prod_{i=1}^{k-1} g_i\left(\tau_{s_i}\right) \mid x_t\right\} Q\left\{g_k(\tau_t) x_t\right\} .
\end{aligned}
$$

Standard arguments show that (3.2.17) holds for all functions $g_1, g_2, \ldots, g_{k-1}$ with support on $]-\infty, t[$ and all functions g_k with support on $[t, \infty[$.

<u>3.3</u> Let $M'(\omega) = \{u\colon\ u = \tau_s(\tilde{\omega})$ for some $s\}$ and $M(\omega)$ be a closure of $M'(\tilde{\omega})$. Note that M' is closed from the right and $M \setminus M'$ consists of no more than a countable number of points. It is obvious that

(3.3.1) $$\tau_s(\tilde{\omega}) = \inf\{t: \ t > s,\ t \in M(\tilde{\omega})\} \ .$$

Let V be a replica of D and x' be the image in V of the point x in D. Put

$$x_t^*(\tilde{\omega}) = \begin{cases} x_t(\omega) & \text{if} \quad t \overline{\in} M(\tilde{\omega}) \\ x_t'(\omega) & \text{if} \quad t \in M(\tilde{\omega}) \end{cases}$$

<u>Lemma 3.3.1</u> For each fixed t

$$Q\{x_t \neq x_t^*\} = 0$$

and the finite dimensional distributions of (x_t^*, Q) are equal to those of $(x_t, \bar{P})$.

<u>Proof</u>.

1) The second statement of the lemma is a trivial consequence of the first one. To Prove the first one it is enough to show that for any fixed t

$$Q\{t \in M\} = 0 \ .$$

Fix $t \in T$. If $t \in M'$ then either $\tau_t = t$ or $\tau_s = t$ for some rational $s < t$. Due to the condition 2.1.A for each fixed t

$$\begin{aligned} 0 &= T_{t-}1(x) - T_t 1(x) \\ &= \lim_{s \uparrow t} \bar{P}_x\{x_s \in D\} - \bar{P}_x\{x_t \in D\} \\ &= \bar{P}_x\{\alpha_{t-} - \alpha_t\} \ . \end{aligned}$$

Consequently, $P_x\{\alpha_{t-} - \alpha_t\} = 0$ and

$$Q\{\tau_s = s + u\} = Q\{Q\{1_{u+s}(\tau_s)|\mathcal{F}\}\} = \bar{P}\{\bar{P}_{x_s}\{\alpha_{u-} - \alpha_u\}\} \quad .$$

Similarly, applying 3.2.β, we get

$$Q\{\tau_t = t\} = \bar{P}\{\lim_{u\downarrow t} \alpha_u^t \neq 1\} = 0 \quad .$$

Therefore, $Q\{t \in M'\} = 0$.

2) If $t \in M\backslash M'$ then $\tau_s < t$ for each rational $s < t$ and vice versa. By 3.2.B

$$Q\{\tau_s \leq t\} = \bar{P}\{1 - \alpha_t^s\} \quad .$$

Therefore,

$$(3.3.2) \qquad Q\{\tau_s < t \text{ for each rational } s < t\} = \bar{P}\{R\text{-}\lim_{s\uparrow t} \alpha_t^s \neq 1\} \quad .$$

Since $\bar{P}$ is a stationary measure and $\alpha_{v+r}^{u+r}(\omega) = \alpha_v^u(\theta_r\omega)$, then the right side of (3.3.2) does not depend on t. Applying the Fubini theorem, we get

$$Q\{t \in M\backslash M'\} = \int_0^1 Q\{s \in M\backslash M'\}ds = Q\left\{\int_0^1 1_s(M\backslash M')ds\right\} = 0 \quad .$$

<u>Lemma 3.3.2</u> The subprocess in D of the process (x_t^*, Q) is equal to $(w(s), P)$.

<u>Proof</u>. It is enough to consider the finite-dimensional distributions of the subprocess, that is, the expressions

$$Q\{x^*_{t_1} \in \Gamma_1, x^*_{t_2} \in \Gamma_2, \ldots, x^*_{t_n} \in \Gamma_n, [t_1,t_n] \cap M = \emptyset\} \quad , \tag{3.3.3}$$
$$t_1 < t_2 < \cdots < t_n \quad ; \quad \Gamma_1, \Gamma_2, \ldots, \Gamma_n \subset D \quad .$$

Put $A_i = \{x_{t_i} \in \Gamma_i\}$, $i = 1, 2, \ldots, n$. By (3.3.1) the expression (3.3.3) equals

$$Q\{A_1A_2 \cdots A_n, \tau_{t_1} > t_n\} = Q\{Q\{A_1 \cdots A_n, \tau_{t_1} > t_n | \mathscr{F}\}\} \;. \tag{3.3.4}$$

By 3.2.B the expression (3.3.4) equals

$$Q\left\{1_{A_1 \cdots A_n} \alpha^{t_1}_{t_n}\right\} = \bar{P}\left\{1_{A_1} \alpha^{t_1}_{t_2} 1_{A_2} \alpha^{t_2}_{t_3} \cdots 1_{A_{n-1}} \alpha^{t_{n-1}}_{t_n} 1_{A_n}\right\} \;. \tag{3.3.5}$$

We know that α^a_b is $\mathscr{F}^a \wedge \mathscr{F}_b$-measurable. Therefore, we may apply the Markov property to the right side of (3.3.5). Doing so, we get

$$Q\left\{A_1, \ldots, A_n, \tau_{t_1} > t_n\right\} = \bar{P}\left\{1_{A_1} \alpha^{t_1}_{t_2} \cdots 1_{A_{n-1}} \varphi(x_{t_{n-1}})\right\} \quad ,$$

where

$$\varphi(x) = \bar{P}_x\left\{1_{A_n} \alpha_{t_n - t_{n-1}}\right\} = p(t_n - t_{n-1}, x; \Gamma_n) \quad ,$$

Repeating this argument $(n - 1)$ times, we get

$$Q\left\{A_1, A_2, \ldots, A_n, \tau_{t_1} > t_n\right\}$$

$$= \int_{\Gamma_1} \nu(dx_1) p(t_2 - t_1, x_1; dx_2) \int_{\Gamma_2} p(t_3 - t_2, x_2; dx_3) \cdots$$

$$\int_{\Gamma_{n-1}} p(t_n - t_{n-1}, x_{n-1}; \Gamma_n) \quad .$$

and that is the finite dimensional distribution of $(w(s),P)$.

<u>3.4</u> Suppose that $\tilde{T}_t$ and $\hat{T}_t$ are two semigroups which are enhancing of T_t. Let $\bar{p}$ and $\hat{p}$ be the transition functions of these semigroups. In Section 3.2 we proved that it is possible to construct Markov processes $(\tilde{y}_t,\tilde{Q})$ and $(\hat{y}_t,\hat{Q})$ in $D + V$ in such a way that the one-dimensional distribution of $(\tilde{y}_t,\tilde{Q})$ (of $(\hat{y}_t,\hat{Q})$) is concentrated on D and is equal to ν and the transition function of $(\tilde{y}_t,\tilde{Q})$ (of $(\hat{y}_t,\hat{Q})$) is equal to $\tilde{p}$(to $\hat{p}$); and both $(\tilde{y}_t,\tilde{Q})$ and $(\hat{y}_t,\hat{Q})$ are covering processes for $(w(s),P)$.

By Theorem 2 of Taksar [1981] the finite dimensional distributions of $(\tilde{y}_t,\tilde{Q})$ coincide with those of $(\hat{y}_t,\hat{Q})$. Therefore, for $\Gamma_1, \Gamma_2 \subset D$ and $t \in T$

$$\int_{\Gamma_1} \nu(dx)\hat{p}(t,x;\ \Gamma_2) = \int_{\Gamma_1} \nu(dx)\tilde{p}(t,x;\ \Gamma_2) \quad . \tag{3.4.1}$$

Consequently

$$\hat{p}(t,x;\ \Gamma_2) = \tilde{p}(t,x;\ \Gamma_2) \quad \text{for } \nu\text{-a.e. } x \quad . \tag{3.4.2}$$

Standard arguments show that (3.4.2) is true simultaneously for all $t \in T$.

It is necessary to mention that in our situation not all the conditions of Theorem 2 of Taksar [1981] hold. The set M does not

satisfy 5.1.A of Taksar [1981] , i.e., it is not progressive measurable with respect to the filtrations generated by the process (x_t^*, Q). Nevertheless, if we replace 5.1.A of Taksar [1981] by the condition 3.4.A below all the proofs remain the same without any changes.

<u>3.4.A</u> For any $\mathscr{F}_t$-measurable random variable ξ, any integer k, and any functions $f_1, f_2, \ldots, f_k$ on T with support on $]-\infty, t[$ and any $s_1 < s_1 < \cdots < s_k$ the random variable $\xi f_1(\tau_{s_1})\, f_2(\tau_{s_2}) \cdots f_k(\tau_{s_k})$ and the random variable τ_t are conditionally independent, given x_t^*.

In our situation 3.4.A is an immediate consequence of Lemma 3.2.3 and Lemma 3.3.1.

The following example shows that we can not prove that $\tilde{T}_t$ in Theorem 2 is unique everywhere (not up to the measure ν).

Let D_1 be a Borel space and $T_t^{(1)}$ be a semigroup such that there exist at least two semigroups R_t^1 and R_t^2, $R_\cdot^1 \neq R_\cdot^2$ and both semigroups are enhancing of $T_t^{(1)}$. (The example of such semigroup $T_t^{(1)}$ was given in Section 1.3 in the paper of Taksar [1981]). Let D_2 be another Borel space and $T_t^{(2)}$ be a semigroup on D_2. Put $D = D_1 \cup D_2$ and consider a semigroup T_t on D defined

$$T_t f(x) = 1_{D_1}(x) T_t^{(1)}(f 1_{D_1})(x) + 1_{D_2}(x) T_t^{(2)}(f 1_{D_2})(x) \quad .$$

We write $T_t = T_t^{(1)} + T_t^{(2)}$. It is easy to see that if $\bar{\nu}$ is a measure on D_2 which is excessive with respect to $T_t^{(2)}$ then the measure ν on D defined

$$\nu(\Gamma) = \bar{\nu}(\Gamma \cap D_2)$$

is excessive with respect to T_t. Moreover, if $\bar{\nu}$ is extreme, then so is ν. Let R_t be a semigroup on D_2 which enhances T_t'' and preserves $\bar{\nu}$. Put $\tilde{T}_t' = R_t^1 + R_t$ and $\tilde{T}_t'' = R_t^2 + R_t$. Then both $\tilde{T}_t'$ and $\tilde{T}_t''$ are enhancing of T_t and ν is an invariant measure with respect to both $\tilde{T}_t'$ and $\tilde{T}_t''$.

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